FOUR-MANIFOLDS WHICH ADMIT $\mathbb{Z}_p \times \mathbb{Z}_p$ ACTIONS

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Abstract. We show that the simply-connected four-manifolds which admit locally linear, homologically trivial $\mathbb{Z}_p \times \mathbb{Z}_p$ actions are homeomorphic to connected sums of $\pm\mathbb{C}P^2$ and $S^2 \times S^2$ (with one exception: pseudofree $\mathbb{Z}_3 \times \mathbb{Z}_3$ actions on the Chern manifold), and also establish an equivariant decomposition theorem.

This generalizes results from a 1970 paper by Orlik and Raymond about torus actions, and complements more recent work of Fintushel, Yoshida, and Huck on S^1 actions. In each case, the simply-connected four-manifolds which support such actions are essentially the same.

1. Introduction

In 1970, Orlik and Raymond [12] proved that any closed, simply connected four-manifold which admits a smooth, effective $S^1 \times S^1$ action can be expressed as a connected sum of copies of $S^2 \times S^2$, $\mathbb{C}P^2$, and $-\mathbb{C}P^2$. Later, Fintushel [4] and Yoshida [16] each showed that the same conclusion holds for smooth S^1 actions. In 1995, Huck [9] generalized this result to show that the intersection form of a closed cohomology four-manifold M with $H_1(M) = 0$ on which S^1 acts must split as a sum of rank 1 and 2 forms (± 1) and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, provided a certain regularity condition holds near the fixed-point set of the action. Huck and Puppe [10] subsequently generalized further by removing the restriction on $H_1(M)$.

Stated simply, Huck's approach is to study the equivariant cohomology of the singular set of an S^1 action using earlier techniques of Puppe [13], and thereby derive a characterization of the possible intersection forms. Related methods were used independently by the author [11] to study actions of finite nonabelian groups on four-manifolds. Our methods actually simplify somewhat when the groups are abelian, and we apply them here to prove: If M is a closed four-manifold with $H_1(M) = 0$ which admits a locally linear, homologically trivial action by $\mathbb{Z}_p \times \mathbb{Z}_p$ (with p prime), then the intersection form of M splits as a sum of copies of (± 1) and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

If this M is simply connected, then by the work of Freedman [5], it must be homeomorphic to a connected sum of copies of $S^2 \times S^2$, $\pm\mathbb{C}P^2$, and perhaps a copy of $\pm\widehat{\mathbb{C}P^2}$, where $\widehat{\mathbb{C}P^2}$ denotes the manifold homotopy equivalent to $\mathbb{C}P^2$, but with non-vanishing Kirby-Siebenmann invariant. We generalize an observation of Wilczyński [15] to show (with exactly one exception) that $KS(M)$ must vanish. As a corollary, we obtain an analogue of Orlik and Raymond's result for $\mathbb{Z}_p \times \mathbb{Z}_p$ actions.

Finally, we discuss the question of classifying the actions themselves. A complete classification would be very difficult, but by combining our results with those of

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Orlik and Raymond, we prove an equivariant version of the decomposition theorem which reduces the general question to that of classifying the possible actions on S^4.

2. The singular set of a $\mathbb{Z}_p \times \mathbb{Z}_p$ action

Suppose M is a closed, connected four-manifold with $H_1(M) = 0$ and $b_2(M) \geq 1$. If $G = \mathbb{Z}_p \times \mathbb{Z}_p$ acts on M, the set $\Sigma = \{x \in M \mid G_x \neq \{0\}\}$ is called the *singular set* of the action. We assume throughout the paper that the action is effective, locally linear, and homologically trivial. By results of Edmonds [2], each $g \neq 0$ in G has a fixed point set consisting of isolated points and 2-spheres, and each 2-sphere represents a nontrivial homology class. Our first task is to understand how the fixed-point sets of the cyclic subgroups $\mathbb{Z}_p \subset G$ fit together to form the overall structure of Σ.

Recall that the Borel equivariant cohomology $H_G(X)$ of a G-space X is the ordinary cohomology of the balanced product $EG \times_G X$. This balanced product has a natural fibration over $EG/G = BG$, and the Leray-Serre spectral sequence of the fibration is called the *Borel spectral sequence*.

The next lemma and the proposition which follows it appeared in slightly different form in Edmonds [2] and [3], but we re-state them here for convenience:

Lemma 2.1. *Suppose G acts homologically trivially on a closed four-manifold M with $H_1(M) = 0$. If either*

1. *$H^2(M)$ contains a class u whose square generates $H^4(M)$, and $H^3(G)$ has no 3-torsion, or*
2. *$b_2(M) \geq 3$,*

then the Borel spectral sequence $E(M)$ collapses with coefficients in $\mathbb{Z}$ or any field. It follows that $H^*_G(M)$ is a free $H^*(G)$ module on $b_2(M)+2$ generators corresponding to generators for $H^*(M)$.

Proof. If $u \in H^2(M)$ has nonzero square, then, since $u^3 = 0$, $0 = d_3(u^3) = 3d_3(u)u^2$. But $E_3^{*,4}$ is a free $H^*(G)$ module generated by u^2. So if $H^3(G)$ has no 3-torsion, then $d_3(u)$ must be 0. And then, of course, $d_3(u^2) = 0$, as well. Thus $E_2(M) = E_3(M)$. Since $d_5(u^2) = 2ud_5(u) = 0$, the sequence collapses.

Now suppose $b_2(M) \geq 3$. Then for each generator $u \in H^2(M)$, there is a $v \in H^2(M)$ which is linearly independent of u in $H^2(M)$, and such that $uv = 0$. Since the action of G is homologically trivial, $E_2(M)$ is a free $H^*(G)$-module on generators corresponding to those of $H^*(M)$, so in fact u and v must be independent in $E_2(M)$, as well. But $d_3(uv) = ud_3(v) + vd_3(u) = 0$. This is only possible if $d_3(u) = d_3(v) = 0$. $H^4(M)$ is generated by products of two-dimensional classes, so $d_3^{*,4} = 0$, as well. It follows that $E_2(M) = E_3(M) = E_4(M)$. The same argument shows that $E_4(M) = E_5(M) = E_\infty(M)$.

The conclusion about $H^*_G(M)$ follows immediately from tom Dieck [14, III.1.18]. □

Whenever G is a finite group and S is a multiplicative, central subset of the cohomology ring $H^*(G)$, we can define the "S-singular set" $\Sigma_S = \{x \in X | S \cap \ker r^*_{G_x} = \emptyset\}$. The fundamental *Localization Theorem* (See Hsiang [8] or tom Dieck [14]) then states that the localized restriction map $S^{-1}H^*_G(M) \to S^{-1}H^*_G(\Sigma_S)$ is an isomorphism. Applying the Localization Theorem in specific cases requires careful choice of S, based on knowledge of the restriction maps from the cohomology of G to that

of its subgroups. But it can yield useful information about the structure of Σ. We apply it to prove:

Proposition 2.2. *Let M be a closed four-manifold such that $H_1(M) = 0$, and suppose that either $b_2(M) \geq 3$, or $b_2(M) = 2$ but the intersection form of M is diagonalizable over $\mathbb{Z}$. If G is a finite abelian group which acts locally linearly and homologically trivially on M, then the rank of G is at most 2, and G has nonempty fixed-point set. If $G = \mathbb{Z}_p \times \mathbb{Z}_p$, then* $\mathrm{Fix}(G)$ *consists of exactly $b_2(M) + 2$ points.*

Proof. Suppose first that $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ acts on M. By Lemma 2.1, $H^*_G(M; \mathbb{Z}_2)$ is a free $H^*(G, \mathbb{Z}_2)$ module on $b_2(M) + 2$ generators. Recall that $H^*(G; \mathbb{Z}_2) \cong \mathbb{Z}_2[a] \otimes \mathbb{Z}_2[b] \otimes \mathbb{Z}_2[c]$, where a, b, and c generate $\mathrm{Hom}(G, \mathbb{Z}_2)$. Since $H^*(G; \mathbb{Z}_2)$ is a polynomial ring, it contains no zero-divisors, so it makes sense to localize at the set S consisting of all of the nonzero elements. We check easily that $S^{-1}H^*(G, \mathbb{Z}_2) \cong \mathbb{Z}_2$.

Now, each proper subgroup $H \subset G$ is the kernel of some nonzero homomorphism $\varphi_H : G \to \mathbb{Z}_2$, and this φ_H, viewed as an element of $H^1(G; \mathbb{Z}_2)$, restricts trivially to $H^1(H; \mathbb{Z}_2)$. So the S-singular set Σ_S contains only those points fixed by all of G. By the Localization Theorem, Σ_S is nonempty. Consideration of the isotropy representation of G at a fixed point x_0 shows that there must be $g, h \in G$ such that g fixes a two-dimensional subspace $V \subset T_{x_0}$, while $h|_V$ acts by $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. But V forms part of a 2-sphere S fixed by g. If h reverses orientation on V, it also acts by -1 on $[S] \in H_2(M)$, contradicting homological triviality.

If $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, the same argument shows that $\mathrm{Fix}(G)$ contains $b_2 + 2$ points, but of course no contradiction ensues from the isotropy representation of G.

If p is odd, a similar argument applies, except in the case where $b_2(M) = 2$ and $p = 3$. To ensure that S is central in $H^*(G; \mathbb{Z}_p)$, we replace the one-dimensional generators of $\mathrm{Hom}(G, \mathbb{Z}_p)$, with their two-dimensional images under the Bockstein map, which generate the polynomial part of $H^*(G, \mathbb{Z}_p)$.

Finally, suppose $b_2(M) = 2$ and $G = \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, with generators g, h, and k. The Lefschetz fixed-point theorem implies that $\chi(\mathrm{Fix}(g)) = 4$. Thus $\mathrm{Fix}(g)$ either contains at least one 2-sphere, or consists of exactly four isolated points. In the first case, $G/\langle g \rangle$ acts effectively on the sphere, which is impossible. In the second case, the action of h on $\mathrm{Fix}(g)$ must fix at least one point x_0. But $\langle g, h \rangle$ cannot act freely on the linking sphere to x_0, so some other element g' fixes a 2-sphere, and the argument proceeds as before.

□

There are indeed actions of $\mathbb{Z}_2 \times \mathbb{Z}_2$ on $S^2 \times S^2$, and pseudofree actions of $\mathbb{Z}_3 \times \mathbb{Z}_3$ on $\mathbb{C}P^2$ and $\widehat{\mathbb{C}P^2}$, whose fixed point set is empty. Inspection shows that such actions are the only exceptions to the rule that $\mathrm{Fix}(G) \neq \emptyset$, and that if $\mathrm{Fix}(G) \neq \emptyset$, then in fact it contains $b_2 + 2$ points. Because our desired conclusion about the intersection form holds in the exceptional cases, we assume as a convenience in this section and the next that the fixed-point set is non-empty. (Later, in the geometrical analysis, the assumption will be more essential, and we will not take it for granted.)

Since the isotropy representation of G at any fixed point splits as a sum of rank two real representations, each fixed point is included in exactly two singular 2-spheres. Since G is abelian, G acts on $\mathrm{Fix}(g)$ for each $g \in G$, so each sphere has a rotation action with fixed points at its north and south poles. Thus Σ contains a total of $b_2 + 2$ spheres $S_1, \ldots, S_{b_2+2}$, and each path component of Σ is a chain of

such spheres arranged in a closed loop. Since the action of G on Σ is just a rotation on each sphere, G acts trivially on $H^*(\Sigma)$.

Lemma 2.3. *If $p=2$, each $[S_i]$ represents a primitive homology class in $H_2(M,\mathbb{Z})$.*

Proof. If each component of Σ contains at least three spheres, then each sphere intersects its neighbor geometrically once, and the claim follows. If some component contains exactly two spheres, then each intersects the other twice. One of them might, a priori, represent a multiple of two in $H_2(M,\mathbb{Z})$. But the theorem of Edmonds cited above implies that it must be nontrivial in $H_2(M;\mathbb{Z}_2)$. □

If p is odd, this argument does not suffice to rule out certain $[S_i]$ being multiples of 2. However, if p is odd, then 2-torsion will not affect the cohomology calculations of the next section. The calculations of that section will show that Σ is connected, and then it will follow that each S_i does, in fact, intersect its neighbor only once.

Our next goal is to show that the inclusion $H_2(\Sigma) \to H_2(M)$ is (split) surjective. When we have shown this, it will follow that the intersection form of M is represented by the geometrical intersections of the spheres in Σ.

From the cohomology long exact sequence of the pair (M,Σ), we extract:

$$0 \to H^1(\Sigma) \to H^2(M,\Sigma) \to H^2(M) \to H^2(\Sigma) \to H^3(M,\Sigma) \to 0.$$

A short diagram chase shows that G acts trivially on the relative cohomology groups. Let N denote the number of path components of Σ, and L, the (integral) rank of $\operatorname{coker} H^1(\Sigma) \to H^2(M,\Sigma)$. As we have noted, each S_i represents an "almost primitive" homology class in M. More precisely: $H^3(M,\Sigma) \cong \mathbb{Z}^{L+2} \oplus T$, where $T=0$ if $p=2$, and $2T=0$ if p is odd.

We shall prove:

Lemma 2.4. *$L=0$,*

From which the claim about $H_2(\Sigma) \to H_2(M)$ is immediate.

Proof. Recall that

$$H^*(\mathbb{Z}_2 \times \mathbb{Z}_2;\mathbb{Z}) \cong \frac{\mathbb{Z}[\alpha_2,\beta_2] \otimes P[\mu_3]}{\langle 2\alpha = 2\beta = 2\mu = 0, \mu^2 = \alpha\beta^2 + \alpha^2\beta \rangle},$$

while for p odd,

$$H^*(\mathbb{Z}_p \times \mathbb{Z}_p;\mathbb{Z}) \cong \frac{\mathbb{Z}[\alpha_2,\beta_2] \otimes E[\mu_3]}{\langle p\alpha = p\beta = p\mu = 0 \rangle}.$$

Let π denote the projection $M \to M/G = M^*$. The Borel spectral sequence of the pair (M,Σ) has

$$E_2^{i,j}(M,\Sigma) = H^i(G; H^j(M,\Sigma)) \Rightarrow H^*_G(M,\Sigma).$$

On the other hand, $M-\Sigma$ is a free G-space, so $H^*_G(M,\Sigma)$ is canonically isomorphic to $H^*(M^*,\Sigma^*)$.

Since (M^*,Σ^*) is a relative manifold pair, Poincaré duality gives a commutative diagram:

$$\begin{array}{ccccc}
 & & H^3(M,\Sigma) & \xrightarrow{\cong} & H_1(M-\Sigma) \\
 & & \uparrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \pi_*} \\
H^3_G(M,\Sigma) & \xrightarrow{\cong} & H^3(M^*,\Sigma^*) & \xrightarrow{\cong} & H_1(M^*-\Sigma^*).
\end{array}$$

But $H_1(M - \Sigma)$ is generated by meridians to the spheres in Σ, and each of these is a p-fold cover of its image in $H_1(M^* - \Sigma^*)$. Thus π_* is multiplication by p. Since the left-hand edge homomorphism $E_2^{0,j} \to E_\infty^{0,j}$ of the Borel spectral sequence is induced by the fiber inclusion $j : (M, \Sigma) \to (M_G, \Sigma_G)$, we can conclude that $\operatorname{coker}(E_2^{0,3} \to E_\infty^{0,3})$ has exponent p. In other words, no $\mathbb{Z}$ summand of $E_2^{0,3}$ supports more than one non-zero differential. In rank counting arguments we can therefore treat its integral rank as though it were a $\mathbb{Z}_p$ rank.

Notice also that, since $H^4(M^*, \Sigma^*) \cong \mathbb{Z}$, each nonzero class in $E_2^{i,j}$ with $i + j \geq 4, i \neq 0$, must be mortal.

Consider the terms of $E_2(M, \Sigma)$ indicated in Table 1:

Table 1. $E_2(M, \Sigma)$

$H^4(M,\Sigma)$	$\mathbb{Z}$	0				
$H^3(M,\Sigma)$	$\mathbb{Z}^{2+L} \oplus T$	0				
$H^2(M,\Sigma)$	$\mathbb{Z}^{N+L}$	0	$\mathbb{Z}_p^{2N+2L}$			
$H^1(M,\Sigma)$	$\mathbb{Z}^{N-1}$	0		$\mathbb{Z}_p^{N-1}$	$\mathbb{Z}_p^{3N-3}$	
0	0	0	0	0	0	0
	$\mathbb{Z}$	0	$\langle \alpha, \beta \rangle$	$\langle \mu \rangle$	$\langle \alpha^2, \alpha\beta, \beta^2 \rangle$	$\langle \mu\alpha, \mu\beta \rangle$

Now, elements of $E_2^{3,1}$ can only be killed by $d_2^{0,2}$, while $E_2^{2,2}$ can be killed either by $d_2^{0,3}$ or $d_2^{2,2}$. By the above observations, we have:

$$\operatorname{rk} E^{0,3} + \operatorname{rk} E^{4,1} \geq \operatorname{rk} E^{2,2} + \operatorname{rk} E^{3,1}$$
$$(2 + L) + (3N - 3) \geq (2N + 2L) + (N - 1),$$

so $L = 0$, as claimed. ☐

We obtain a corollary which is dual, in some sense, to Edmonds's theorem [2, 2.5]:

Corollary 2.5. *Suppose $G = \mathbb{Z}_p \times \mathbb{Z}_p$ acts as we have been assuming. Then the cohomology restriction map $H^2(M) \to H^2(\Sigma)$ is injective, so Σ represents all of $H_2(M)$.*

To better understand the geometry of the singular set, we also prove:

Lemma 2.6. $N = 1$*, so Σ is connected.*

Proof. From the homology spectral sequence of the covering $\pi : X - \Sigma \to X^* - \Sigma^*$, we obtain a short exact sequence $0 \to \mathbb{Z} \times \mathbb{Z} \times T \xrightarrow{\pi_*} H_1(M^* - \Sigma^*) \to \mathbb{Z}_p \times \mathbb{Z}_p \to 0$. As we have already seen, π_* is multiplication by p in each factor. It follows that $H_1(M^* - \Sigma^*) \cong H^3(M^*, \Sigma^*)$ is p-torsion-free. So in fact all classes in $E_2^{i,j}$ with $i > 0$ are mortal. In particular, $E_2^{1,2}$ must vanish, so $N \geq 2(N - 2)$, so $N \leq 2$. Suppose for a contradiction that $N = 2$. The Borel spectral sequence then takes the following form (each entry with $i > 0$ is $\mathbb{Z}_p^k$ for some k, so to save space, we simply indicate its rank):

There are generators $a \in H^1(M, \Sigma)$ and $b, c \in H^2(M, \Sigma)$ such that $d_2(b) = \alpha a$ and $d_2(c) = \beta a$. By the multiplicative properties of the spectral sequence, this kills the entire row $j = 1$, *except* $H^3(G; H^1(M, \Sigma) \cong \mathbb{Z}_p \cong \langle \mu \rangle$.

Now, $\ker d_2^{2,2} = \langle \beta b - \alpha c \rangle$, so there is some $e \in H^3(M, \Sigma)$ such that $d_2(e) = \beta b - \alpha c$. Since $E_3^{3,1} = \langle \mu a \rangle$ must also perish, there is $f \in H^3(M, \Sigma)$, independent

Table 2. $E_2(M,\Sigma)$

$H^4(M,\Sigma)$	$\mathbb{Z}$	0	2	1	3	2
$H^3(M,\Sigma)$	$\mathbb{Z}^2 \oplus T$	0	4	2	6	4
$H^2(M,\Sigma)$	$\mathbb{Z}^2$	0	4	2	6	4
$H^1(M,\Sigma)$	$\mathbb{Z}$	0	2	1	3	2
0	0	0	0	0	0	0
	$\mathbb{Z}$	0	$\langle \alpha, \beta \rangle$	$\langle \mu \rangle$	$\langle \alpha^2, \alpha\beta, \beta^2 \rangle$	$\langle \mu\alpha, \mu\beta \rangle$

of e, so that $d_3(f) = \mu a$. But then $d_3(\alpha f) = d_3(\beta f) = 0$, since $\mu\alpha a$ and $\mu\beta b$ were already killed by d_2. Now $\ker d_2^{2,3}$ has rank 2, and $d_3^{2,3} = 0$. But $d_2^{0,4}$ has rank ≤ 1, so $E_\infty^{2,3}$ must have rank ≥ 1. This is a contradiction, so $N = 1$. $\square$

Now that we know this, each S_i definitely intersects each neighbor only once, so $T = 0$ for odd p.

To summarize, we have shown:

Proposition 2.7. *Suppose M is a closed, topological four-manifold with $b_2(M) \geq 1$ and $H_1(M) = 0$, equipped with an effective, homologically trivial, locally linear $\mathbb{Z}_p \times \mathbb{Z}_p$ action. With the exception of fixed-point free actions which exist in the two cases,*

1. *$b_2(M) = 1$, $p = 3$, and the action is pseudofree, or*
2. *M has intersection form $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ when $p = 2$,*

the singular set Σ consists of $b_2(M) + 2$ spheres equipped with rotation actions, intersecting pairwise at their poles, and arranged into a single closed loop. Each sphere represents a primitive class in $H_2(M;\mathbb{Z})$, and together these classes generate $H_2(M)$.

3. The intersection form

Let $\sigma_1, \ldots, \sigma_{b_2+2}$ denote the fundamental classes of $S_1, \ldots, S_{b_2+2} \in \Sigma$. As generators of $H_2(M)$, two can be regarded as "redundant". If we eliminate one, we cut the loop of Σ. By removing another, we either disconnect or shorten the remaining chain. Renumber the remaining spheres, if necessary, as $S_1, \ldots, S_{b_2}$, and call the result Σ'. Let $e_i = \sigma_i \cdot \sigma_i$. The matrix of intersections of the spheres, and therefore the intersection form of M, as well, takes the form of one, or a sum of two, pieces of the form

$$\begin{pmatrix} e_1 & 1 & & & & & \\ 1 & e_2 & 1 & & & & \\ & 1 & e_3 & 1 & & & \\ & & 1 & \ddots & & & \\ & & & & & 1 & \\ & & & & 1 & e_{k-1} & 1 \\ & & & & & 1 & e_k \end{pmatrix}$$

Huck and Yoshida have already proven exactly the lemma we need about such matrices (See Huck [9, lemma 4.2]): Each is equivalent to a sum of rank 1 and 2 pieces. Thus:

Theorem 3.1. *Suppose M is a closed topological four-manifold with $H_1(M) = 0$. Let p be prime, and suppose $\mathbb{Z}_p \times \mathbb{Z}_p$ acts effectively, locally linearly, and homologically trivially on M. Then the intersection form of M is a sum of copies of (± 1) and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.*

4. Vanishing of $KS(M)$

Edmonds [1] showed that when p is a prime greater than 3, locally linear, homologically trivial $\mathbb{Z}_p$ actions exist on every simply-connected four-manifold. The actions he constructs are *pseudofree* – i.e. with only isolated fixed points. In certain cases, this is a necessary restriction. For example, Wilczyński [15] shows that if a homotopy $\mathbb{C}P^2$ admits a $\mathbb{Z}_p$ action which fixes a two-sphere, then the two-sphere can be used to split off $\mathbb{C}P^2$ as a connected summand of M, and it follows that M is homeomorphic to $\mathbb{C}P^2$. In other words, $\widehat{\mathbb{C}P^2}$ admits *only* pseudofree actions.

For the remainder of the paper, we assume M is simply connected. We generalize Wilczyński's construction to prove the following corollary of Theorem 3.1:

Theorem 4.1. *If M is a closed, simply connected four-manifold which admits a locally linear, homologically trivial $\mathbb{Z}_p \times \mathbb{Z}_p$ action, then M is homeomorphic to a connected sum of copies of $\pm\mathbb{C}P^2$ and $S^2 \times S^2$, or if $p = 3$ and the action is pseudofree, perhaps to a single copy of $\pm\widehat{\mathbb{C}P^2}$.*

(In Theorem 5.2, we will establish a sharper result.)

Proof. For convenience, we continue to assume the action is not pseudofree, or fixed-point-free in the case of $S^2 \times S^2$. As above, let Σ' denote the singular set with two homologically redundant spheres removed. For simplicity of notation, we assume Σ' is connected; if it isn't, our argument will carry through on each piece.

It follows from the work of Freedman and Quinn [6, 9.3] (see also [7, 1.2]) that each $S^2 \subset \Sigma$ has an equivariant normal bundle. Thus Σ' has a regular neighborhood $N(\Sigma')$ which is homeomorphic to the manifold obtained by plumbing together disk bundles $E(e_i)$, $i = 1, \ldots, b_2$, over S^2 according to the graph

$$A_{b_2} = e_1 \frac{\qquad}{} e_2 \frac{\qquad}{} \cdots \frac{\qquad}{} e_{b_2 - 1} \frac{\qquad}{} e_{b_2}.$$

The boundary of such a plumbed manifold is a lens space L, and $|H_1(L)|$ is given by the determinant of the intersection matrix. In our case, the matrix is unimodular, so L is in fact a three-sphere. Thus $M' = N(\Sigma') \cup_{S^3} D^4$ is homeomorphic to a connected sum of copies of $\pm\mathbb{C}P^2$ and $S^2 \times S^2$. But M' is also a connected summand of M which carries all of its homology. By Freedman and Quinn [6, 10.3], $M' \cong M$. □

5. On classifying $\mathbb{Z}_p \times \mathbb{Z}_p$ actions

The argument of Orlik and Raymond on the classification of torus actions, specialized to the simply-connected case, can be summarized as follows: The quotient space M/T is a surface with boundary, and since $H_1(M) = 0$, it must be a disk. The boundary of D consists of fixed points and arcs; the arcs can be labeled according to the corresponding isotropy subgroups of T. Each arc lifts to a singular S^2 and each interior point of the disk represents a principal orbit. They show that the quotient map in fact admits an essentially unique section; thus the singular data in the quotient space determine M up to equivariant diffeomorphism. A calculation

involving the particular isotropy groups then shows that the quotient space splits in a way which lifts to an equivariant connected sum decomposition of M.

Up to an automorphism of T, listing the one-dimensional isotropy groups is equivalent to listing the Euler classes and (signed) intersection numbers of the singular 2-spheres. This information is also available for $\mathbb{Z}_p \times \mathbb{Z}_p$ actions. To what extent does it classify them? We will show:

Proposition 5.1. *Assume the action is not one of the exceptional fixed-point-free cases.*

1. *Each $\mathbb{Z}_p \times \mathbb{Z}_p$ action extends to a torus action in a regular neighborhood $\nu(\Sigma)$ of Σ.*
2. *$\nu(\Sigma)$ is T-equivariantly diffeomorphic to the singular set of some smooth T-action on M, but the given T-action need not extend over M.*

Proof. We begin with a slight variant of the plumbing construction of the previous section: Let $t = b_2(M) + 2$, and label the spheres consecutively around Σ as $S_1, \dots, S_t$. Let x_i denote the "north pole" of S_i. Choose orientations for each of the S_i, and let σ_i denote the corresponding fundamental class. Finally, choose an orientation for M and let $\epsilon_i = \sigma_{i-1} \cdot \sigma_i$ denote the sign of the intersection at x_i. (When we considered Σ' earlier, we implicitly chose orientations to make each $\epsilon_i = +1$; here, because the spheres are arranged in a closed loop, this might not be possible.)

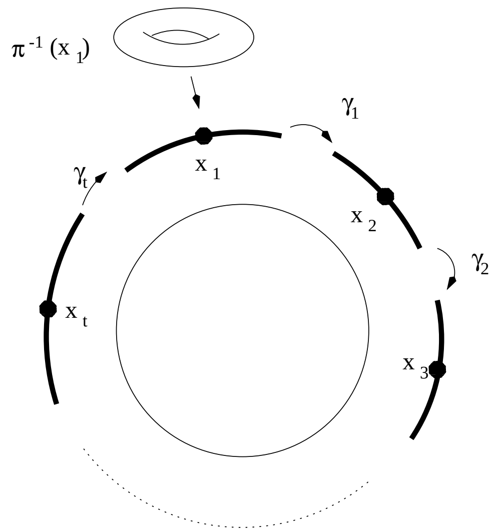
With these conventions, $\nu(\Sigma)$ is obtained by plumbing together D^2-bundles ξ_i over S^2, each with Euler class e_i, according to the orientations given by the ϵ_i. The plumbing graph is a circle, which we parameterize as $[\frac{1}{2}, t + \frac{1}{2}]$ with the endpoints identified. $\partial\nu$ can be thought of as a torus fiber bundle over the plumbing graph with a fiber-preserving, free $\mathbb{Z}_p \times \mathbb{Z}_p$ action. It is not a priori a principal bundle, but if the $\mathbb{Z}_p \times \mathbb{Z}_p$ action on the fibers extends to a torus action, it will become one. With appropriate smoothing around the plumbing points, the torus action will extend over $\nu(\Sigma)$, establishing the first part of the proposition.

The T-bundle over $[\frac{1}{2}, t+\frac{1}{2}]$ can be assembled by gluing copies of $T \times [i-\frac{1}{2}, i+\frac{1}{2}]$ via attaching maps γ_i which incorporate the clutching functions for the ξ_i, the coordinate switches at each plumbing point, and the orientations ϵ_i. (See figure 1, which is intended to invite comparison with the diagrams in [12].) The maps are determined up to isotopy by their $\pi_1(T)$ representations. The clutching functions take the form $\left(\begin{smallmatrix} 1 & 0 \\ e_i & 1 \end{smallmatrix}\right)$; the coordinate switches are of course $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and the orientation changes, $\left(\begin{smallmatrix} 1 & 0 \\ 0 & \epsilon_i \end{smallmatrix}\right)$. Together, such matrices generate $GL(2,\mathbb{Z})$, the structure group of the bundle.

The $\mathbb{Z}_p \times \mathbb{Z}_p$ action has well-defined rotation numbers in the fiber over x_1, so it extends to a torus action in that fiber. The gluing maps in $GL(2,\mathbb{Z})$ define a trivialization of the bundle over $T \times [\frac{1}{2}, t + \frac{1}{2}]$ which is equivariant with respect to the $\mathbb{Z}_p \times \mathbb{Z}_p$ action. Using them, the torus action extends along all of the fibers. The structure of the torus bundle is thus determined by the total gluing function $\gamma : T \times \{t + \frac{1}{2}\} \to T \times \{\frac{1}{2}\}$; with slight abuse of notation, we may write $\gamma = \gamma_t \circ \cdots \circ \gamma_2 \circ \gamma_1$.

A compatibility condition is imposed by the existence of the $\mathbb{Z}_p \times \mathbb{Z}_p$ action – namely, that the gluing map $\gamma \in GL(2,\mathbb{Z})$ must commute with order p rotations in each factor of $T \times 0$. We may analyze this requirement by lifting to the universal cover $\pi : \widetilde{T} \to T$. A rotation r lifts to a translation τ. The requirement that $\pi_*(\gamma^{-1}\tau\gamma) = \pi_*(\tau) = r$ means that the line spanned by each τ must be

FIGURE 1. Torus bundle over the plumbing graph



1. Normalized by γ, if $p = 2$. Since the total space of the bundle is the boundary of $\nu(\Sigma)$, it is orientable, so γ is one of $\pm\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ or $\pm\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$,
2. Centralized by γ, if $p > 2$, which implies $\gamma = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$.

In the latter case, γ clearly commutes with the entire torus action, so $\partial\nu$ supports the structure of a principal bundle. Even when $p = 2$, γ must respect the base-fiber splitting of the bundle ξ_t over S_t, so $\pm\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$ is ruled out. We proceed to rule out $\gamma = \left(\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right)$, also. If such a bundle is realized on the singular set of a $\mathbb{Z}_2 \times \mathbb{Z}_2$ action, let μ be a small meridional loop around S_t in $M - \Sigma$. Then μ is homologous to $-\mu$, and so $2\mu = 0$ in $H_1(M - \Sigma)$. But $H_1(M - \Sigma) \cong H^3(M, M - \Sigma)$ is torsion-free, as we saw in section 2. It is generated by any pair of meridians to neighboring two-spheres in Σ.

This finishes the proof that $\partial\nu$ is a trivial principal T-bundle, and hence also the proof of the first claim. Our proof of the second claim is constructive, based on Orlik and Raymond's model in the case of torus actions.

The orbit space $A = \nu(\Sigma)/T$ is an annulus. Its outer boundary component $\partial_1 A$ consists of t fixed points separated by t arcs whose stabilizers are copies of $S^1 \subset T$. Its inner boundary $\partial_2 A$ consists entirely of principal orbits. Adjoin a disk D to $\partial_2 A$. Because the torus bundle is trivial over $\partial_2 A$, there is no obstruction to lifting this adjunction to a T-equivariant gluing of $D^2 \times T$ to $\partial\nu$. The resulting manifold, denoted M', is simply connected and has the same intersection form as M, so it is homeomorphic to M.

Finally, an example of Hambleton, Lee, and Madsen ([7]) shows that M and M' need not be $\mathbb{Z}_p \times \mathbb{Z}_p$-equivariantly homeomorphic. They begin with a linear $\mathbb{Z}_p \times \mathbb{Z}_p$ action on $\mathbb{C}P^2$, and equivariantly connect sum a $\mathbb{Z}_p$-orbit of counterexamples to the Smith conjecture in S^4 around one of the singular 2-spheres. The resulting space is still homeomorphic to $\mathbb{C}P^2$, but the complement of the singular set has nonabelian fundamental group. In the linear example, $\mathbb{C}P^2 - \Sigma$ has the homotopy type of a torus. $\square$

Let us call a $\mathbb{Z}_p \times \mathbb{Z}_p$ action *standard* if it is the restriction of a smooth torus action. It is fair to say that the standard actions are completely understood. Proposition 5.1, together with the construction of section 4, shows that we can equivariantly split off standard summands. If the two "redundant" two-spheres are adjacent, then $M \cong M_{\text{standard}} \# S^4$, while if there is no such choice of adjacent spheres, a two-step splitting still yields $M \cong M'_{\text{standard}} \# M''_{\text{standard}} \# S^4$. Because the standard actions extend to torus actions, Orlik and Raymond's classification theorem applies to show that each splits further into "irreducible" pieces. This proves:

Theorem 5.2. *Let M be a closed, simply-connected four-manifold with an effective, locally linear, homologically trivial $\mathbb{Z}_p \times \mathbb{Z}_p$ action. Assume the action is not one of the fixed-point-free exceptions. Then M admits an equivariant connected sum decomposition*

$$M \cong S^4 \# M_1 \# \ldots \# M_k,$$

where each M_i is one of S^4, $S^2 \times S^2$, $\pm\mathbb{C}P^2$, or $\mathbb{C}P^2 \# -\mathbb{C}P^2$, equipped with a standard action. The action on the first S^4 summand need not be standard.

Recall that the "fixed-point-free exceptions" are the pseudofree actions of $\mathbb{Z}_3 \times \mathbb{Z}_3$ on $\mathbb{C}P^2$ and $\widehat{\mathbb{C}P^2}$, and fixed-point-free actions of $\mathbb{Z}_2 \times \mathbb{Z}_2$ on $S^2 \times S^2$. Also note that Orlik and Raymond construct examples of torus actions on $\mathbb{C}P^2 \# -\mathbb{C}P^2$ which admit no *equivariant* connected sum decomposition.

As a consequence of this theorem, the general problem of classifying $\mathbb{Z}_p \times \mathbb{Z}_p$ actions on simply-connected four-manifolds reduces to the question of classifying actions on S^4. The latter is still, of course, very difficult.

6. Questions

Finally, we leave the reader with two questions:

1. What can constructively be said about the classification of $\mathbb{Z}_p \times \mathbb{Z}_p$ actions on S^4, in light of the possible knotting of the singular set?
2. Huck and Puppe [10] generalized Huck's earlier work on circle actions to the case $H_1(M) \neq 0$. Does Theorem 3.1 generalize similarly? It is worth noting that in the general case, the singular set need not contain spheres, as examples of free actions on T^4 easily show.

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